

Quantum circuits of T -depth one

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We give a Clifford+ T representation of the Toffoli gate of T -depth 1, using four ancillas. More generally, we describe a class of circuits whose T -depth can be reduced to 1 by using sufficiently many ancillas. We show that the cost of adding an additional control to any controlled gate is at most 8 additional T -gates, and T -depth 2. We also show that the circuit THT does not possess a T -depth 1 representation with an arbitrary number of ancillas initialized to $|0\rangle$.

1 Introduction

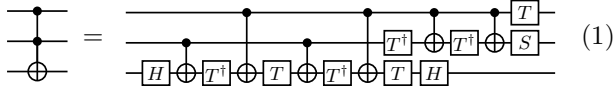
It is known that the gates of the Clifford group, together with the single-qubit non-Clifford gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix},$$

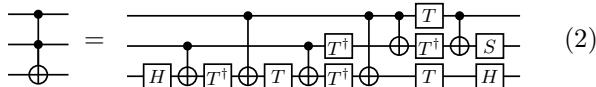
form a good universal gate set for fault-tolerant quantum computation [2]. The decomposition of arbitrary gates into this Clifford+ T set, either exactly or to within some given accuracy ϵ , is an important problem [3]. It is often desirable to find decompositions that are optimal with respect to a given cost function. The exact cost function used is application dependent; some possibilities are: the total number of gates; the total number of T -gates; the circuit depth; and/or the number of ancillas used.

Amy et al. [1] recently proposed T -depth as a cost function. The idea is to count the number of T -stages in a circuit, rather than the number of T -gates. A T -stage is a group of one or more T - and/or T^\dagger -gates on distinct qubits that can be performed simultaneously. Note that, for the purpose of computing T -count or T -depth, the gates T and T^\dagger can be treated interchangeably, due to the identity $T^\dagger = TS^\dagger$.

To illustrate the concept of T -depth, consider the standard decomposition of the Toffoli gate into the Clifford+ T set, as given in [4]:

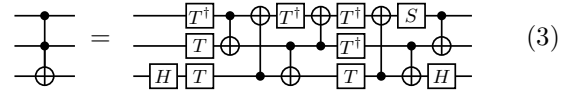


This decomposition has T -count 7, and in the exact form written, it has T -depth 6, because the fourth and fifth T -gates form a single T -stage. Using trivial commutations, the circuit (1) can easily be reduced to T -depth 4:



Amy et al. [1] further improved the T -depth of the Toffoli gate to 3, using the following circuit. They conjecture

that for circuits without ancillas, this T -depth is optimal.



The purpose of this note is to show that, with the use of ancillas, the T -depth of the Toffoli gate, and of many (but not all) other circuits, can be reduced to 1. This may be useful in quantum computing architectures where T -gates are expensive and ancillas are cheap.

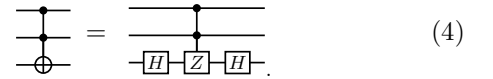
2 A T -depth one representation of the Toffoli gate

Recall that the Clifford group for any number of qubits is generated by the Hadamard gate H , the phase gate $S = T^2$, the controlled not-gate, and unit scalars. As usual, we write X , Y , and Z for the Pauli operators.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

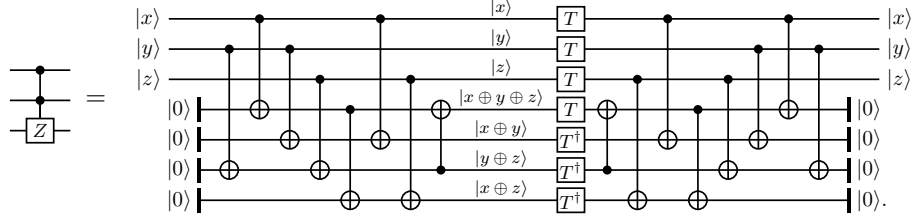
The Toffoli gate is a doubly-controlled not-gate. It is equivalent to a doubly-controlled Z -gate via a basis change:



Now consider a computational basis state $|xyz\rangle$, where $x, y, z \in \{0, 1\}$. The effect of the doubly-controlled Z -gate is to map $|xyz\rangle$ to $(-1)^{xyz}|xyz\rangle$. Let us write “ \oplus ” for modulo-2 addition in $\{0, 1\}$, and “+” and “−” for the usual addition and subtraction of integers. We then have the following inclusion-exclusion style formula for $x, y, z \in \{0, 1\}$:

$$4xyz = x + y + z - (x \oplus y) - (y \oplus z) - (x \oplus z) + (x \oplus y \oplus z). \quad (5)$$

This is easy to prove by case distinction, or algebraically using $x \oplus y = x + y - 2xy$. Now let $\omega = (-1)^{1/4} = e^{i\pi/4}$.



From (5), we have

Note that $T|x\rangle = \omega^x|x\rangle$, and therefore, the doubly-controlled Z -gate can be implemented by applying T -gates to qubits in states $|x\rangle$, $|y\rangle$, $|z\rangle$, and $|x \oplus y \oplus z\rangle$, and T^\dagger -gates to qubits in states $|x \oplus y\rangle$, $|y \oplus z\rangle$, and $|x \oplus z\rangle$. This can be done in any order, or even in parallel, using four ancillas, as shown in Figure 1. Combining this with (4), we obtain a representation of the Toffoli gate of T -depth 1 and overall depth 7.

3 An application to multiply-controlled gates

The double-controlled Z -gate is a diagonal gate whose effect is given by (6). The controlled S^\dagger -gate is a diagonal gate whose effect is given by $(-i)^{xy} = (\omega^\dagger)^x (\omega^\dagger)^y \omega^{x \oplus y}$. It follows that the combined effect of the two gates is

which can therefore be implemented with T -count 4. Using one ancilla, this can be done with T -depth 1 and overall depth 5:

Alternatively, one can find an implementation that uses no ancilla. It uses fewer overall gates, but has T -depth 2 and over overall depth 7:

Let us write

and we use the mirror image notation to denote the inverse of this gate. Suppose we have a Clifford+ T -representation of some controlled quantum gate G , and we wish to obtain an efficient Clifford+ T -representation of a doubly-controlled G -gate. Using (9), (11), and (12), the cost of doing so is at most 8 additional T -gates, increasing the T -depth by at most 2, and the overall depth by at most 14, using 2 ancillas:

Note that the cost of the additional control, in terms of the overall gate count, is 28 (2 times 12 gates from (9) and 2 times 2 Hadamard gates from (11)). This can be reduced to 26 by leaving the ancilla in (9) in state $|x\rangle$ instead of $|0\rangle$; however, doing so requires carrying this ancilla during the computation of G , which may involve a tradeoff.

shown here for $n = 3$:

$$(13)$$

For example, this yields an implementation of a triply-controlled not-gate with T -count 15 and T -depth 3 (7 T -gates for the Toffoli gate, and 8 T -gates for the additional control); or a quintuply-controlled not-gate with T -count 31 and T -depth 5. It is not currently known whether any of these T -counts or depths are optimal.

Remark 3.2. Because the T -gate is diagonal with $T|0\rangle = |0\rangle$, it can be regarded as a controlled gate, namely, a controlled global phase change. Therefore, we can use the above procedure to implement a controlled T -gate with T -count 9 as follows:

$$(14)$$

Using (9), we obtain T -depth 3, depth 15, and gate count 29 with two ancillas. As before, by leaving the ancilla of (9) in state $|x\rangle$ instead of $|0\rangle$, the gate count can be reduced to 27. Alternatively, using (10), we obtain T -depth 5, depth 19, and gate count 27 with one ancilla. Except for slightly improved overall gate counts, these results are the same as those in [1].

4 T -depth one representation of almost classical circuits

It is straightforward to generalize the construction of Section 2 to circuits built up from T and *almost classical* gates.

Definition 4.1. A unitary operator is *classical* if it is given by a permutation of computational basis states, and *diagonal* if its matrix representation is diagonal in the computational basis. Let us call an operator *almost classical* if it can be written as a product of a classical operator and a diagonal operator.

The almost classical operators obviously form a group. Of the 24 single-qubit Clifford operators (taken modulo global phase), exactly 8 are almost classical; they form the subgroup generated by S and X .

Definition 4.2. Let \mathcal{C} be a set of gates. We say that a circuit is $\mathcal{C} + T$ -representable if it can be built with gates from $\mathcal{C} \cup \{T\}$ and their inverses. We say that such a circuit has T -depth n (relative to \mathcal{C}) if it can be written using only gates from \mathcal{C} and n T -stages.

Theorem 4.3. Let \mathcal{C} be any set of almost classical gates, containing the controlled not-gate. Using ancillas, any $\mathcal{C} + T$ -representable n -qubit circuit can be written of T -depth 1 (relative to \mathcal{C}).

Proof. The proof idea is simple. Each T -gate in the circuit is a $\pi/4$ phase change conditioned on some boolean combination of the inputs. Intuitively, one may copy each such boolean condition to an ancilla, execute all T -gates in parallel, uncompute the ancillas, and finally re-compute the output.

The formal proof proceeds by induction on circuits. For each $\mathcal{C} + T$ -representable n -qubit circuit A , we will by induction construct $\mathcal{C} + T$ -representable circuits A_1 and A_2 such that A_1 is diagonal and has T -depth at most 1, A_2 has T -depth 0, and $A = A_2 \circ A_1$.

The base case occurs when $A = I$ is the identity circuit. In this case, we can let $A_1 = A_2 = I$, and there is nothing to show.

For the induction step, suppose A is of the form $A' \circ G$, where G is a single gate. By induction hypothesis, there is a decomposition $A' = A'_2 \circ A'_1$ satisfying the above conditions.

- Case 1: G is not equal to T or T^\dagger . In this case, we let $A_1 = G^\dagger \circ A'_1 \circ G$ and $A_2 = A'_2 \circ G$. Then trivially, $A = A_2 \circ A_1$, and A_1 and A_2 have the required T -depths. Moreover, since G is almost classical, A_1 is diagonal.
- Case 2: G is T , applied to the i th qubit. In this case, we let

$$A_1 = (15)$$

and $A_2 = A'_2$. Since A'_1 is diagonal, so is A_1 , and it follows that the ancilla is uncomputed correctly. Moreover, A_1 is equivalent to $A'_1 \circ G$, and therefore, $A = A_2 \circ A_1$. Finally, since A'_1 has T -depth at most 1, so does A_1 .

- Case 3: G is T^\dagger , applied to the i th qubit. This is entirely analogous to case 2. \square

Remark 4.4. The gate set \mathcal{C} in Theorem 4.3 is not necessarily assumed to consist of Clifford gates. For example, if on some hypothetical architecture, T -gates are expensive but Toffoli gates are cheap, one can include the Toffoli gate in the set \mathcal{C} .

Remark 4.5. In general, the proof of Theorem 4.3 increases the size of the circuit, but only by a constant factor. In practice, it is often possible to find a much smaller circuit than the one constructed in the proof.

Remark 4.6. Taking $\mathcal{C} = \{S, X, CNOT\}$ and applying Theorem 4.3 to circuit (1) (excluding the initial and final Hadamard gate) yields another T -depth one representation of the Toffoli gate.

Remark 4.7. There is a trade-off between T -depth and the number of ancillas. The procedure of the proof of Theorem 4.3 adds one ancilla for each T -gate. However, by splitting a circuit with T -count n into two circuits with T -count $\lceil n/2 \rceil$ each, it is clear that one can approximately half the number of ancillas by doubling the T -depth, and so forth.

Remark 4.8. Version 2 of [1], which appeared following the private communication credited as [24] therein, contains a similar result in Section 6.4, but with a proof that is quite different.

5 Some circuits cannot be written with T -depth one

The result of the previous section shows that any two T -stages can be combined into a single T -stage, provided that they are only separated by almost classical gates. One may wonder whether perhaps *all* Clifford+ T circuits can be written of T -depth one, using a sufficient number of ancillas initialized to $|0\rangle$. We show that this cannot be done.

Theorem 5.1. *The single-qubit operator THT cannot be implemented as a Clifford+ T circuit of T -depth 1, using an arbitrary number of ancillas initialized to $|0\rangle$. This is true even if the ancillas are not required to be returned to their initial state at the end of the computation.*

Before proving the theorem, we start with a general observation about Clifford+ T circuits of T -depth 1.

Proposition 5.2. *Let U be an n -qubit Clifford+ T circuit of T -depth 1. Let $|\phi\rangle$ be any single-qubit state, and consider*

$$|\psi\rangle = U(|\phi\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle).$$

Consider the $\{+1, -1\}$ -valued Pauli observable X applied to the first qubit of ψ ; denote its expected value by $E_{|\phi\rangle}$. Suppose $E_{|+\rangle}$ is non-zero. Then

$$\frac{E_{|0\rangle}}{E_{|+\rangle}}$$

is a rational number.

Proof. The expected value of the observable X on the first qubit of $|\psi\rangle$ is

$$\begin{aligned} E_{|\phi\rangle} &= \langle \psi | (X \otimes I \otimes \dots \otimes I) | \psi \rangle \\ &= \langle \phi, 0, \dots, 0 | U^\dagger (X \otimes I \otimes \dots \otimes I) U | \phi, 0, \dots, 0 \rangle. \end{aligned} \quad (16)$$

We analyze the structure of $U^\dagger (X \otimes I \otimes \dots \otimes I) U$. Since U is of T -depth 1, it can be written as $U = U_3 \circ U_2 \circ U_1$, where U_1 and U_3 are Clifford circuits and $U_2 = T \otimes \dots \otimes T \otimes I \otimes \dots \otimes I$. Since U_1 is Clifford, we know that $U_1^\dagger (X \otimes I \otimes \dots \otimes I) U_1$ is a Pauli operator

$$U_1^\dagger (X \otimes I \otimes \dots \otimes I) U_1 = \pm A_1 \otimes \dots \otimes A_n, \quad (17)$$

where each $A_i \in \{X, Y, Z, I\}$. Using the relations

$$\begin{aligned} T^\dagger I T &= I, & T^\dagger Z T &= Z, \\ T^\dagger X T &= \frac{1}{\sqrt{2}}X - \frac{1}{\sqrt{2}}Y, & T^\dagger Y T &= \frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y, \end{aligned}$$

we find that

$$\begin{aligned} &U_2^\dagger (\pm A_1 \otimes \dots \otimes A_n) U_2 \\ &= \pm (T^\dagger A_1 T) \otimes \dots \otimes (T^\dagger A_n T) \otimes A_{n+1} \otimes \dots \otimes A_n \\ &= \lambda P_1 + \lambda P_2 + \dots + \lambda P_m, \end{aligned} \quad (18)$$

where each P_j is an n -qubit Pauli operator. The key observation here is that the *same* factor λ occurs in front of each (possibly signed) summand, and λ is independent of $|\phi\rangle$. In fact, we have $\lambda = (\frac{1}{\sqrt{2}})^k$, where k is the number of times the operators X and Y occur among A_1, \dots, A_n . Let

$$Q_j = U_3^\dagger P_j U_3. \quad (19)$$

Since U_3 is Clifford, this is again some Pauli operator, say

$$Q_j = (-1)^{q_j} B_{j,1} \otimes \dots \otimes B_{j,n}. \quad (20)$$

Combining (17) through (20), we find

$$\begin{aligned} U^\dagger (X \otimes I \otimes \dots \otimes I) U &= \lambda Q_1 + \lambda Q_2 + \dots + \lambda Q_m \\ &= \lambda \sum_{j=1}^m (-1)^{q_j} B_{j,1} \otimes \dots \otimes B_{j,n}. \end{aligned} \quad (21)$$

Combining this with (16), we get

$$E_{|\phi\rangle} = \lambda \sum_{j=1}^m (-1)^{q_j} \langle \phi | B_{j,1} | \phi \rangle \langle 0 | B_{j,2} | 0 \rangle \dots \langle 0 | B_{j,n} | 0 \rangle. \quad (22)$$

Since each $B_{j,i} \in \{X, Y, Z, I\}$ is a Pauli operator, it follows that $E_{|\phi\rangle}/\lambda$ is rational (indeed, an integer) for $|\phi\rangle \in \{|0\rangle, |+\rangle\}$. The claim then immediately follows. \square

Proof of Theorem 5.1. For $U = THT$, we compute

$$U^\dagger X U = \frac{1}{2}X + \frac{1}{2}Y + \frac{1}{\sqrt{2}}Z,$$

and therefore

$$E_{|0\rangle} = \langle 0 | U^\dagger X U | 0 \rangle = \frac{1}{\sqrt{2}}$$

and

$$E_{|+\rangle} = \langle + | U^\dagger X U | + \rangle = \frac{1}{2}.$$

Since $E_{|0\rangle}/E_{|+\rangle}$ is irrational, the claim immediately follows from Proposition 5.2. \square

6 Acknowledgements

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References

- [1] M. Amy, D. Maslov, M. Mosca, and M. Roetteler. A meet-in-the-middle algorithm for fast synthesis of depth-optimal quantum circuits. arXiv:1206.0758, 2012.
- [2] H. Buhrman, R. Cleve, M. Laurent, N. Linden, A. Schrijver, and F. Unger. New limits on fault-tolerant quantum computation. In *Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2006)*, pages 411–419, 2006.
- [3] V. Kliuchnikov, D. Maslov, and M. Mosca. Fast and efficient exact synthesis of single qubit unitaries generated by Clifford and T gates. arXiv:1206.5236v2, June 2012.
- [4] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2002.